Sains Malaysiana 40(6)(2011): 643–650

Incorporating Optimisation Technique into Zadeh's Extension Principle for Computing Non-Monotone Functions with Fuzzy Variable

(Menggabungkan Teknik Pengoptimuman ke dalam Prinsip Perluasan Zadeh untuk Komputeran Fungsi-Fungsi Tak Bermonoton dengan Pembolehubah Kabur)

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ABSTRACT

This paper proposes a new computational method for computing non-monotone functions that take a fuzzy interval as their arguments. The proposed method represents an implementation of optimisation technique into Zadeh's extension principle. By taking into account the dependency problem that exists in fuzzy environment, the proposed method can avoid the effect of overestimation in computation. This problem usually arises when the same fuzzy interval is computed separately in fuzzy interval computation. The proposed method is simple to use and can be implemented in many practical applications. In order to show the capability of the proposed method, several non-monotone functions with trapezoidal fuzzy intervals are studied.

Keywords: Fuzzy set; optimisation; Zadeh's extension principle

ABSTRAK

Makalah ini mencadangkan satu kaedah komputasi baru untuk komputeran fungsi-fungsi tak bermonoton yang mengambil selang kabur sebagai pembolehubahnya. Kaedah yang dicadangkan merupakan suatu perlaksanaan teknik pengoptimuman di dalam prinsip perluasan Zadeh. Dengan mengambilkira masalah kebergantungan yang wujud di dalam persekitaran kabur, kaedah yang dicadangkan ini dapat mengelakkan masalah terlebih anggaran dalam pengiraan. Masalah ini biasanya wujud apabila selang kabur yang sama dikira secara berasingan di dalam komputasi selang kabur. Kaedah ini mudah untuk dilaksanakan dan dapat diterapkan di dalam pelbagai penggunaan praktikal. Untuk menunjukkan kebolehan kaedah yang dicadangkan, beberapa fungsi tak bermonoton dengan selang kabur trapezoid dikaji.

Kata kunci: Pengoptimuman; prinsip perluasan Zadeh; set kabur

INTRODUCTION

The mathematics of fuzzy set theory was coined in 1965 by Zadeh (1965). Since its birth, the theory of fuzzy set has been rigorously developed and it has influenced in many fields of application. For example, it has been extensively used in control system, image processing, communication and integrated circuit manufacturing. One of the main fundamental principles in fuzzy set theory is the so called Zadeh's extension principle (Zadeh 1965). It provides a mechanism of extending a real continuous function to a function accepting fuzzy set as its argument. In general, the computation of Zadeh's extension principle is rather difficult tasks. The simplicity can only be found if the function to be extended is monotone. However, the difficulty arises when the function is non-monotone (Chalco-Cano et al. 2009). Without a proper method, the computation of Zadeh's extension principle may not guarantee to have low computational complexity since it would require infinite numbers of computation.

Today, several methods have been proposed in order to compute Zadeh's extension principle. One of the earlier methods was proposed by Kaufmann and Gupta (1991), which based on the α – cuts and interval arithmetic. However, the results are not completely satisfying. In fact, the use of the straightforward interval arithmetic into computation leads to overestimation in the results. Due to this, many researchers have proposed some new techniques such as the requisite constraint (Klir 1997), the fuzzy weighted average (Dong & Wong 1987; Wood et al. 1992 and Yang et al. 1993), the vertex method (Dong & Shah 1987), the transformation method (Hanss 2002 & Klimke 2003), and the spline approximation method (Chalco-Cano et al. 2009). However, these proposed methods increased computational complexity when applied to non-monotone functions as well. Therefore, a new computational method has to be proposed so that the computational complexity and overestimation in the results can be reduced.

In this paper, we propose a new method for computing non-monotone functions that take fuzzy set as their arguments. This method is based on minimising and maximising of a function, which is finding the minimum and maximum at every level of α – cut. In this paper, we only consider the problem of finding the minimum, since the maximum can be easily found by noting that max $g(x)$

= – min (–g(x)), that is the maximum of g(x) is the negative of the minimum of –g(x).

In order to solve the optimisation problems, we use Brent's method (Brent 2002), which combines the golden section search with parabolic interpolation. One of the advantages of using this method is that it does not require the calculation of derivative. This is particularly useful when the derivative of a function, required by most non-monotone functions, is difficult or impossible to obtain analytically. The idea of Brent's method is to find a minimum of a parabola through three points. If the function to be minimised is nicely parabolic near to the minimum, then the parabola fitted through any three points in a single leap to the minimum. In the worst possible case, where the parabolic interpolation is acceptable but useless, then the method will approximately alternate between parabolic interpolation and golden section search (Press et al. 2007).

In case where the function is reduced to monotonically increasing or decreasing, then we find the minimum at the endpoints. These enormous varieties of geometry should be considered to reduce function evaluations during computation. Please note that Brent's method will only find a local minimum and not a global minimum, unless the function is unimodal. By a unimodal function we mean there exists a unique number $m \in [a, b]$ such that the function g(x) is monotonically decreasing on $[a, m]$ and monotonically increasing on $[m, b]$. Even though the method of Simulated Annealing (Kirkpatrick et al. 1983) has been developed to find global minimum, but it is not a practical way for computing Zadeh's extension principle. The reason is that the method requires a big computational effort during iteration and at the end we cannot guarantee that the global minimum found is the correct one.

Basic Concepts

In the following, we briefly elaborate some definitions and important concepts in fuzzy sets theory.

FUZZY SETS

According to Zadeh (1965), a fuzzy set is a generalisation of a classical set that allows membership function to take any value in the unit interval [0, 1]. The formal definition of a fuzzy set is as follow:

Definition 1: Let U be a universal set. A fuzzy set A in U is defined by a membership function $A(x)$ that maps every element in U to the unit interval [0, 1].

A fuzzy set A in U may also be presented as a set of ordered pairs of a generic element x and its membership value, as shown in the following equation:

$$A = \{x, A(x)) \mid x \in U\}. \quad (1)$$

Definition 2: Let A be a fuzzy set defined in U. The support of A is the crisp set of all elements in U such that the membership function of A is non-zero, that is,

$$\text{supp}\,(A) = \{x \in \mathrm{U} \mid A(x) > 0\}. \quad (2)$$

Definition 3: Let A be a fuzzy set defined in U. The core of A is the crisp set of all elements in U such that the membership value of A is 1, that is,

$$\text{core}\,(A) = \{x \in U \mid A(x) = 1\}. \quad (3)$$

Definition 4: Let A be a fuzzy set defined in $\Re$. A is called a fuzzy interval if

1. A is normal, that is there exists $x_0 \in \Re$ such that $A(x_0) = 1$;
2. A is convex, that is for all $x, y \in \Re$ and $0 \le \lambda \le 1$, it holds that

$$A(\lambda x + (1 - \lambda)y) \ge \min(A(x), A(y));$$

3. A is upper semi-continuous, that is for any $x_0 \in \Re$. it holds that

$$A(x_0) \ge \lim_{x \to x_0^{\pm}} A(x);$$

4. $[A]^0 = \overline{\{x \in \Re \mid A(x) \ge \alpha\}}$ is a compact subset of $\Re$.

Definition 5: Let A be a fuzzy interval defined in $\Re$. The α – cut of A is the crisp set $[A]^\alpha$ that contains all elements in $\Re$ such that the membership values of A is greater than or equal to α, that is

$$[A]^\alpha = \{x \in \Re \mid A(x) \ge \alpha\}, \quad \alpha \in (0.1]. \quad (4)$$

For a fuzzy interval A, its α – cuts are closed intervals in $\Re$ and we denote them by

$$[A]^\alpha = [a_1^\alpha, a_2^\alpha], \quad \alpha \in (0, 1]. \quad (5)$$

Definition 6: A fuzzy interval A is called a trapezoidal fuzzy interval if its membership function has the following form:

$$A(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & c \le x \le c, \\ \dfrac{d-x}{d-c}, & c \le x \le d, \\ 0, & x > d, \end{cases} \quad (6)$$

and its α – cuts are simply

$$[A]^\alpha = [a + \alpha(b - a), d - \alpha(d - c)], \quad \alpha \in (0, 1]. \quad (7)$$

This definition asserts that the trapezoidal fuzzy interval A is defined by four numbers $a < b < c < d$, where the core of A is the interval $[b, c]$ and its support is the interval (a, d) Figure 1 shows the example of trapezoidal fuzzy interval.

In this paper the set of all trapezoidal fuzzy intervals will be denoted by $F(\Re)$.

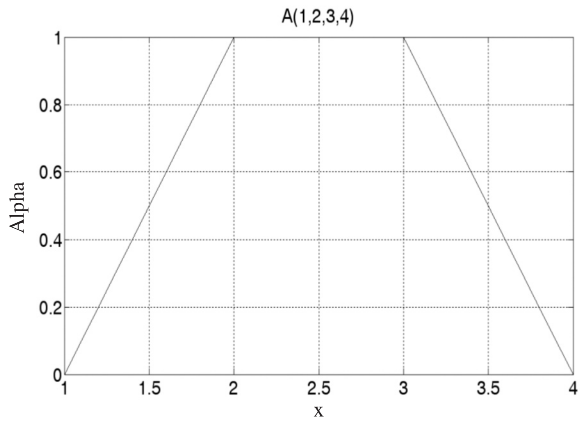

FIGURE 1. Trapezoidal fuzzy interval

ARITHMETIC OPERATIONS OF FUZZY INTERVAL

In this subsection, we recall some arithmetic operations of fuzzy interval. Arithmetic operations of fuzzy interval are generalisation of the operations of interval arithmetic introduced by Moore (1966). First, we recall the four basic arithmetic operations of real interval, namely:

1. addition: $[a, b] + [c, d] = [a + c, b + d]$;
2. subtraction: $[a, b] - [c, d] = [a - d, b + c]$;
3. multiplication:

$$[a, b].[c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)];$$

4. division:

$$\frac{[a,b]}{[c,d]} = \left[\min\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right), \max\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right)\right],$$

where c and $d \neq 0$.

Let A and B are two different fuzzy intervals and denote '*' be any of the four basic arithmetic operations. For $\alpha \in (0,1]$, $[A]^\alpha$ and $[B]^\alpha$ are close intervals in $\Re$ (by definition). Hence, the basic arithmetic operations of fuzzy intervals A and B can be defined as follows:

$$[A*B]^\alpha = [A]^\alpha * [B]^\alpha. \tag{8}$$

In this case the four standard arithmetic operations of real interval can be used directly for every level of α. For division of two different fuzzy intervals, we require that $0 \notin [B]^0$. However, if A and B are the same fuzzy intervals, then the basic arithmetic operations are defined in different ways (see Klir 1997).

THE EXTENSION PRINCIPLE

The idea of the extension principle is easy to understand. Let g be a function that maps from X to Y. The extension principle provides a mechanism to transform a fuzzy set defined in X to a fuzzy set defined in Y.

Let $F(X)$ and $F(Y)$ be the sets of all fuzzy sets defined in X and Y, respectively and $g : X \to Y$ be a continuous function. The function g induces a mapping $g : F(X) \to F(Y)$ such that if A is a fuzzy set in X, then its range under g is a fuzzy set $B = g(A)$ whose membership function is expressed as in the following equation (Zadeh 1975a, Zadeh 1975b and Zadeh 1975c):

$$g(A)(y) = \begin{cases} \sup\limits_{x \in g^{-1}(y)} A(x), & \text{if } y \in range(g), \\ 0 & , \text{if } y \notin range(g), \end{cases} \tag{9}$$

where

$$g^{-1}(y) = \{x \in X \mid g(x) = y\}.$$

Román-Flores et al. (2001) have shown that if $g : X \to Y$ is a real continuous function, then $g : F(X) \to F(Y)$ is a well-defined function, and

$$[g(A)]^\alpha = g([A]^\alpha), \tag{10}$$

for all $\alpha \in [0.1]$ and $A \in F(X)$.

In general, to find a fuzzy set B in Y is not an easy task. An exception occurs when g is monotone. If g is non-monotone, the function values at the endpoints of fuzzy set A in X are not the correct endpoints of fuzzy set B in Y.

THE PROPOSED METHOD

In this section, we first study the concept of dependency problem that exists in fuzzy environment. Then, we present an example with different types of calculations for the same problem. Following this concept, we develop a new method for computing continuous functions that take trapezoidal fuzzy interval as its argument. The computational complexity of the proposed method is also studied.

THE DEPENDENCY PROBLEM

The dependency problem in fuzzy environment exists when the same fuzzy interval is computed separately in fuzzy interval computation. To understand this concept, we give an example. Given the trapezoidal fuzzy interval $A(-1,0,1,2)$ with α – cuts are $[A]^\alpha = [\alpha-1, 2-\alpha]$ for $\alpha \in (0,1]$. Suppose we use the function defined by $g(x) = 5x^2-2x+2$ and we want to find $g(A)$, where g is a real continuous function and A is a trapezoidal fuzzy interval. There are two common ways to find $g(A)$. First, we apply the straightforward fuzzy interval arithmetic, which is based on the α – cut of fuzzy interval:

$$g([A]^\alpha) = 5[\alpha-1, 2-\alpha].[\alpha-1, 2-\alpha]-2[\alpha-1, 2-\alpha]+2$$

If $\alpha = 0$, then we have:

$$g([A]^0) = [-12, 24]. \tag{11}$$

Second, we apply Zadeh's extension principle by considering that the trapezoidal fuzzy interval is computed separately:

$$g(A) = 5A^2 - 2A + 2. \quad (12)$$

For particular $\alpha = 0$, the solution is therefore

$$g([A]^0) = [-2, 24]. \quad (13)$$

Unfortunately, both solutions (see Eqs. (11) and (13)) are not correct because they do not represent the actual range of $g(x)$ for $x \in [-1, 2]$. To get the correct result, we have to define the whole expression on the right hand side of $g(x)$ as a new function. Then we apply Zadeh's extension principle. We refer to the example discussed above and apply this idea to it then we have the following result:

$$g([A]^0) = [9/5, 18], \quad (14)$$

which is the actual range of $g(x)$ for $x \in [-1. 2]$. This dependency problem can also be seen in numerical methods for differential equations with fuzzy initial values. However, many researchers did not take into account this problem when deriving the numerical methods for differential equations with fuzzy initial values (see Ma et al. 1999, Abbasbandy & Allahviranloo 2002, Abbasbandy & Allahviranloo 2004, Pederson & Sambandham 2007, Palligkinis et al. 2008 and Pederson & Sambandham 2008). Consequently, the diameters of the solutions of differential equations with fuzzy initial values increase as t increases. This is always the case when the same fuzzy interval is computed separately in fuzzy interval computation. This is shown in preliminary studies conducted by Ahmad and Hasan (2010).

DISCRETISATION OF TRAPEZOIDAL FUZZY INTERVALS

Let $A(a,b,c,d)$ be a trapezoidal fuzzy interval with α – cuts are denoted by $[A]^\alpha = [a_1^\alpha, a_2^\alpha]$ for all $\alpha \in (0,1]$, where $a_1^\alpha = a + \alpha(b-a)$ and $a_2^\alpha = d - \alpha(d-c)$. First, we descretise α up to n points on the interval (0,1]. The points are equally spaced using $\Delta h = 1/(n-1)$. The discretisation points are given by $\alpha_i = \alpha_{i-1} + \Delta h$, for $i = 2, \ldots, n$. After discretisation, we have the following set of α with n elements of point:

$$\alpha = \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n\}, \quad (15)$$

where $\alpha_1 = 0$, $\alpha_i = \alpha_{i-1} + \Delta h$ and $\alpha_n = 1$ for $i = 2, \ldots, n$. From Eq. (15) and using Definition 5, we have the following set of intervals:

$$I = \{[A]^{\alpha_1}, \ldots, [A]^{\alpha_i}, \ldots, [A]^{\alpha_n}\}. \quad (16)$$

For the different α – cuts of A the following property holds:

$$[A]^{\alpha_{i+1}} \subseteq [A]^{\alpha_i}, \ \forall \alpha_i, \alpha_{i+1} \in (0,1] \text{ with } \alpha_i \le \alpha_{i+1} \quad (17)$$

for $i = 1,2,\ldots,n-1$. From (17), it is clear that the α – cut of A at α_{i+1} is subset of the α – cut of A at α_i (see Figure 2).

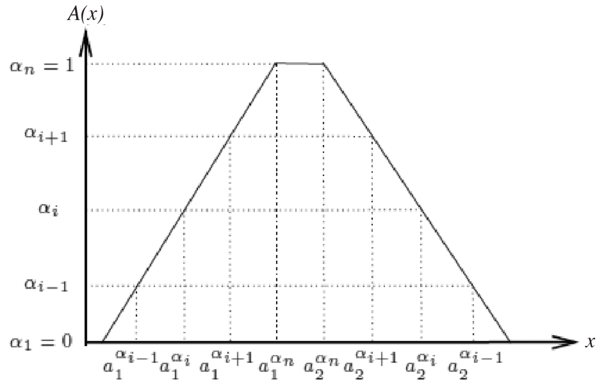

FIGURE 2. α – discretisation of a trapezoidal fuzzy interval

Due to this, the α – cut of A can also be constructed as the union of sub-intervals as shown in the following equations:

$$[A]^{\alpha_n} = \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right] \quad (18)$$

$$\vdots$$

$$[A]^{\alpha_i} = \left[a_1^{\alpha_i}, a_1^{\alpha_{i+1}}\right] \cup \underbrace{\ldots \cup \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right] \cup \ldots}_{[A]^{\alpha_{i+1}}} \cup \left[a_2^{\alpha_{i+1}}, a_2^{\alpha_i}\right], \quad (19)$$

$$\vdots$$

$$[A]^{\alpha_1} = \left[a_1^{\alpha_1}, a_1^{\alpha_2}\right] \cup \underbrace{\ldots \cup \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right] \cup \ldots}_{[A]^{\alpha_{i+1}}} \cup \left[a_2^{\alpha_2}, a_2^{\alpha_1}\right], \quad (20)$$

Let $g : \Re \to \Re$ be a real continuous function and we want to find a trapezoidal fuzzy interval $B = g(A)$ that is induced by g. In this study, we compute $B = g(A)$ at each level of α_i for $i = 1,2, \ldots ,n$ according to the following equations:

$$\hat{b}_1^{\alpha_n} = \min_{x \in \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right]} g(x), \quad (21)$$

$$\hat{b}_2^{\alpha_n} = \max_{x \in \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right]} g(x), \quad (22)$$

$$\vdots$$

$$\hat{b}_1^{\alpha_i} = \min\left[\min_{x \in \left[a_1^{\alpha_i}, a_1^{\alpha_{i+1}}\right]} g(x), \ldots, \min_{x \in \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right]} g(x), \ldots, \min_{x \in \left[a_2^{\alpha_{i+1}}, a_2^{\alpha_i}\right]} g(x)\right], \quad (23)$$

$$\hat{b}_2^{\alpha_i} = \max\left[\max_{x \in \left[a_1^{\alpha_i}, a_1^{\alpha_{i+1}}\right]} g(x), \ldots, \max_{x \in \left[a_1^{\alpha_n}, a_2^{\alpha_n}\right]} g(x), \ldots,\right.$$

$$\left. \max_{x\in\left[a_2^{\alpha_{i+1}},a_2^{\alpha_i}\right]} g(x)\right], \tag{24}$$

$$\vdots$$

$$\hat{b}_1^{\alpha_1} = \min\left[\min_{x\in\left[a_1^{\alpha_1},a_1^{\alpha_2}\right]} g(x),\ldots, \min_{x\in\left[a_1^{\alpha_n},a_2^{\alpha_n}\right]} g(x),\ldots, \min_{x\in\left[a_2^{\alpha_2},a_2^{\alpha_1}\right]} g(x)\right], \tag{25}$$

$$\hat{b}_2^{\alpha_1} = \max\left[\max_{x\in\left[a_1^{\alpha_1},a_1^{\alpha_2}\right]} g(x),\ldots, \max_{x\in\left[a_1^{\alpha_n},a_2^{\alpha_n}\right]} g(x),\ldots, \max_{x\in\left[a_2^{\alpha_2},a_2^{\alpha_1}\right]} g(x)\right], \tag{26}$$

Here, $\hat{b}_1^{\alpha_i}$ and $\hat{b}_2^{\alpha_i}$ are the minimum and maximum values which obtained from Eqs. (21) – (26), which finally turn out to be the endpoints of the α – cuts of trapezoidal fuzzy interval B. The optimisation problems in Eqs. (21) till (26) will be performed as follows: (1) if $g(x)$ is decreasing or increasing on the sub-intervals, then the optimal solutions are obtained at the endpoints of the sub-intervals; (2) if $g(x)$ is unimodal on the sub-intervals, then we use Brent's method (Brent 2002). To test whether $g(x)$ is decreasing, increasing or unimodal on the sub-intervals, we do monotonicity testing. In this test, we take any three points in the sub-intervals. For example, we take a, b and c as the three points in the interval $[a, c]$. These three points are more then enough because the interval $[a, c]$ is a very small interval. Here, a is the lower bound of $[a, c]$, b is the midpoint in the interval $[a, c]$ and c is the upper bound of the interval $[a, c]$. In monotonicity testing, we have the following five possibilities:

for every $a < b < c$,

1. if $g(a) < g(b) < g(c)$, then g is increasing on the interval $[a, c]$. So, the minimum is $g(a)$ and the maximum is $g(c)$;
2. if $g(a) > g(b) > g(c)$, then g is decreasing on the interval $[a, c]$. So, the minimum is $g(c)$ and the maximum is $g(a)$;
3. if $g(a) > g(b) < g(c)$, then g is unimodal on the interval $[a, c]$. So, the minimum is predicted around $g(b)$ and the maximum is max $(g(a),g(c))$;
4. if $g(a) < g(b) > g(c)$, then g is also unimodal on the interval $[a, c]$. So, the minimum is min $(g(a),g(c))$ and the maximum is predicted around $g(b)$; or
5. if $| g(a) - g(b) | < \varepsilon$ and $| g(b) - g(c) | < \varepsilon$, then g is closely horizontal. So, the minimum of $g(x)$ is closely equals to the maximum of $g(x)$ for $x \in [a,c]$.

In order to have low computational complexity, we propose a new strategy to find the minimum and maximum values on the interval $\left[a_1^{\alpha_i}, a_2^{\alpha_i}\right]$. We start from $\alpha_n = 1$ and continue downward until $\alpha_1 = 0$. For instance, at α_i for $i = 1,2,\ldots,n$, we have $2 \cdot (2i - 1)$ optimisation problems to be solved (see Eqs. 23 and 24). However, we only consider the first and the last optimisation problems. The other optimisation problems have already been solved at α_{i+1}. By taking the minimum (maximum) of all results of the optimisation problems, we have a new minimum value (a new maximum value). The minimum value (maximum value) at α_i can be similar to or smaller (bigger) than the minimum (maximum) found at α_{i+1}, depending on the function under consideration. This process is repeated for all levels of α. As a result, we have a set of intervals, which finally turns out to be a trapezoidal fuzzy interval as well. Next, we introduce the following error of computing $B = g(A)$:

Definition 7: Let $g : \Re \to \Re$ be a real continuous function. Given a trapezoidal fuzzy interval A on $\Re$. The error of computing $B = g(A)$ is given by

$$E^{\alpha_i} = \left|b_1^{\alpha_i} - \hat{b}_2^{\alpha_i}\right| + \left|b_2^{\alpha_i} - \hat{b}_2^{\alpha_i}\right|, \; i = 1,2,\ldots,n, \tag{27}$$

where $[B]^{\alpha_i} = \left[b_1^{\alpha_i}, b_2^{\alpha_i}\right]$ and $[\hat{B}]^{\alpha_i} = \left[\hat{b}_1^{\alpha_i}, \hat{b}_2^{\alpha_i}\right]$ are the α – cuts of analytical solution and approximation solution, respectively.

COMPUTATIONAL COMPLEXICITY

The computational complexity of the proposed method can be determined by calculating the total number of function evaluations. It is also depending on the total number of $\alpha \in [0,1]$ that we have discretised. If $g(x)$ is decreasing or increasing on the sub-intervals, then the computational complexity can be calculated as follow:

$$cp = 3 + 6(n - 1), \tag{28}$$

where n is the total number of $\alpha \in [0,1]$.

NUMERICAL EXAMPLES

In this section, we use the proposed method to illustrate the approximation of Zadeh's extension principle for some non-monotone functions. Please note that one requirement for Zadeh's extension principle is that the functions chosen should be continuous on its domain. If the function is one-to-one mapping, the solution of Zadeh's extension principle is straightforward. However, if the function is not one-to-one mapping, the problem arises when two or more distinct points in its domain are mapped into the same point in its image. In this case, we need to take the supremum (maximum) of two or more membership values (Zadeh 1975a, Zadeh 1975b and Zadeh 1975c).

Example 1: We consider the following trapezoidal fuzzy interval A defined by

$$A(x)=\begin{cases}0 & , \quad x<1\\ \frac{5}{4}x-\frac{5}{4} & , \quad 1\le x\le 9/5\\ 1 & , \quad 9/5\le x\le 11/5\\ -\frac{5}{4}+\frac{15}{4} & , \quad 11/5\le x\le 3\\ 0 & , \quad x>3\end{cases}$$

The α – cut of A is given by:

$$[A]^{\alpha} = [4\alpha/5+1, 3-4\alpha/5], \quad \alpha \in (0,1].$$

Suppose we use the following function:

$$g(x) = 3x - x^2$$

and we want to find $g(A) = 3A - A^2$. The function g is continuous on the support of A and it has an extreme point at $x = 1.5$. The analytical solution of Zadeh's extension principle is given by:

$$g(A)(y)=\begin{cases}0, & y<0\\ \frac{5}{8}\left(3-\sqrt{9-4y}\right), & 0\le y\le 1.76\\ 1, & 1.76\le y\le 2\\ \max\left(1,\frac{5}{8}\left(3-\sqrt{9-4y}\right)\right), & 2\le y\le 2.16\\ \max\left(\frac{5}{8}\left(3-\sqrt{9-4y}\right),\frac{5}{8}\left(1+\sqrt{9-4y}\right)\right), & 2.16\le y\le 2.25\\ 0, & y>2.25\end{cases}$$

By using the method proposed in this paper, we obtain the approximation of $g(A)$, which is exactly the same as the analytical solution (see Figure 3(c)). The approximation errors are listed in Table 1. The graphs of $A(x)$, $g(x)$ and $g(A)$ are depicted in Figures 3(a), 3(b) and 3(c), respectively.

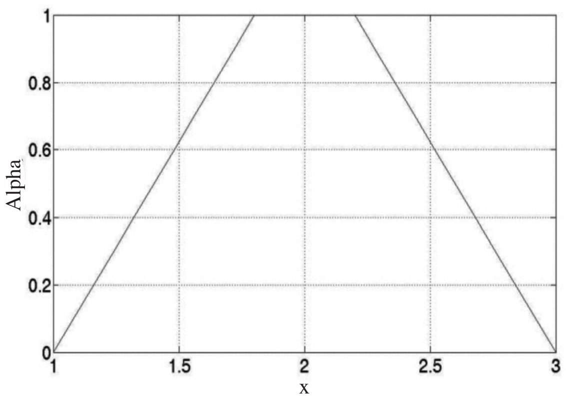

FIGURE 3(a). Fuzzy interval A

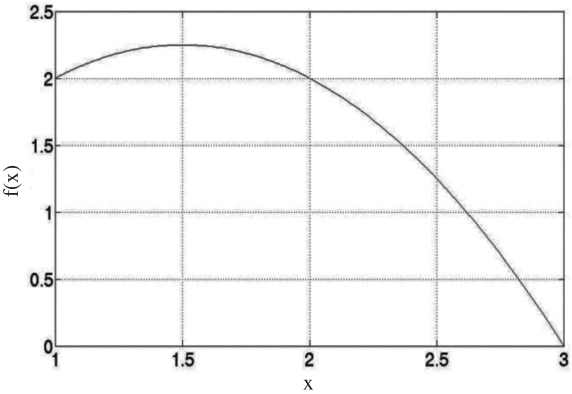

FIGURE 3(b). Function handle

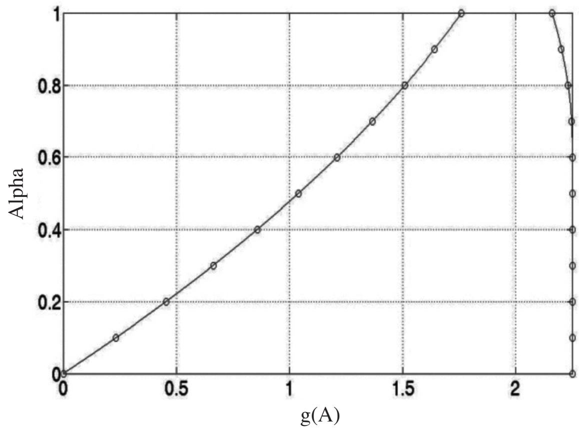

FIGURE 3(c). Comparison between analytical solution (solid line) and its approximation (circle mark)

TABLE 1. Approximation errors for Example 1

α	Error
0.0	0.0E - 05
0.1	0.0E - 05
0.2	0.0E - 05
0.3	0.0E - 05
0.4	0.0E - 05
0.5	0.0E - 05
0.6	0.0E - 05
0.7	0.0E - 05
0.8	0.0E - 05
0.9	0.0E - 05
1.0	0.0E - 05

In term of computational complexity, we observed that the total number of function evaluations required in this example is 63 with $n = 11$.

Example 2: We consider the following trapezoidal fuzzy interval A defined by:

$$A(x)=\begin{cases}0 & , \quad x<\pi/4\\ \dfrac{3}{3\pi-2}\left(4x-\dfrac{\pi}{3\pi-2}\right) & , \quad \pi/4\le x\le(3\pi-1)/6\\ 1 & , \quad (3\pi-1)/6\le x\le(3\pi+1)/6\\ -\dfrac{3}{3\pi-2}\left(4x-\dfrac{3\pi}{3\pi-2}\right), & \quad (3\pi+1)/6\le x\le 3\pi/4\\ 0 & , \quad x>3\pi/4\end{cases}$$

The α – cut of A is given by:

$$[A]^{\alpha}=\left[\frac{(3\pi-2)\alpha+3\pi}{12},\frac{-(3\pi-2)\alpha+9\pi}{12}\right],\alpha\in(0,1].$$

Suppose we use the following function:

$g(x) = \sin(x)$

and we want to find $g(A) = \sin(A)$. The sine function is periodic with a period of 2π. Since it is defined on the support of A, then it has an extreme point at $x = \pi/2$. So, the correct range of $g(A)$ is defined on the interval $\left[1/\sqrt{2},1\right]$. From Zadeh's extension principle, the analytical solution is given by:

$$g(A)(y)=\begin{cases}0. & y<1/\sqrt{2}\\ \max\left(\dfrac{12}{3\pi-2}\sin^{-1}(y)-\dfrac{3\pi}{3\pi-2},\right. & \\ \left.-\dfrac{12}{3\pi-2}\left(\pi-\sin^{-1}(y)\right)+\dfrac{9\pi}{3\pi-2}\right), & 1/\sqrt{2}\le y\le 0.9861\\ 1, & 0.9861\le y\le 1\\ 0, & y>1\end{cases}$$

By using the technique proposed in this paper, we obtain the approximation of $g(A)$. The graphs of $A(x)$, $g(x)$ and $g(A)$ are depicted in Figures 4(a), 4(b) and 4(c), respectively. From the graph, we can see that the approximation solution is exactly equal to the analytical solution. The approximation errors are listed in Table 2.

In term of computational complexity, we observed that the total number of function evaluations required in this example is 70 with $n = 11$.

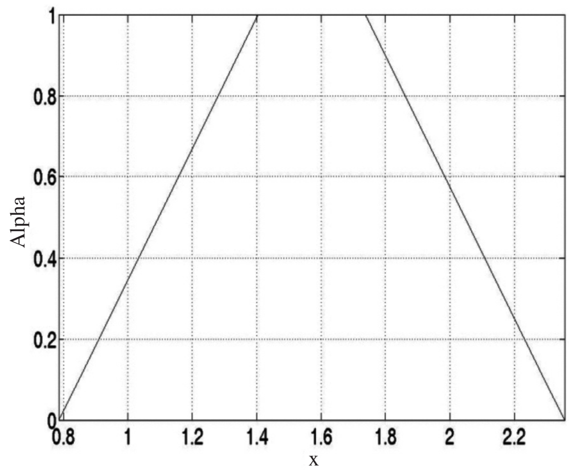

FIGURE 4(a). Fuzzy interval A

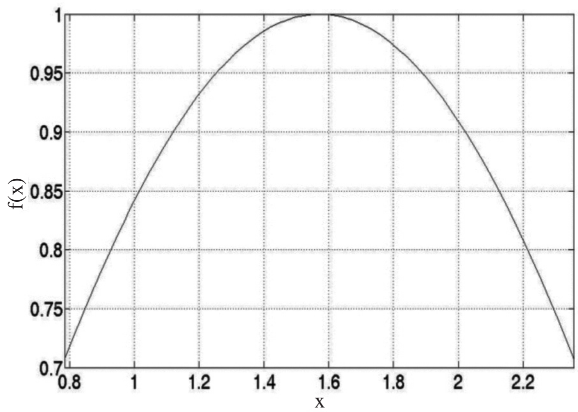

FIGURE 4(b). Function handle

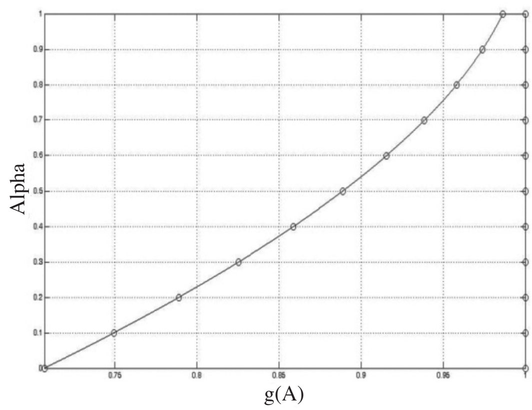

FIGURE 4(c). Comparison between analytical solution (solid line) and its approximation (circle mark)

CONCLUSIONS

We have proposed a new computational method for computing non-monotone functions that take a trapezoidal fuzzy interval as their arguments. The proposed method is based on minimising and maximising of a function, which is the function takes on the minimum and maximum values. In this paper, we have considered only the problem of finding the minimum, since the maximum can be easily found by noting that $\max g(x) = -\min(-g(x))$. The method proposed in this paper greatly improves the computational aspect, especially in handling non-monotone functions. In the future, the proposed method will be incorporated into classical numerical methods for solving non-linear differential equations with fuzzy initial values.

TABLE 2. Approximation errors for Example 2

α	Error
0.0	0.0E - 05
0.1	0.0E - 05
0.2	0.0E - 05
0.3	0.0E - 05
0.4	0.0E - 05
0.5	0.0E - 05
0.6	0.0E - 05
0.7	0.0E - 05
0.8	0.0E - 05
0.9	0.0E - 05
1.0	0.0E - 05

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Received: 23 February 2010
Accepted: 1 June 2010